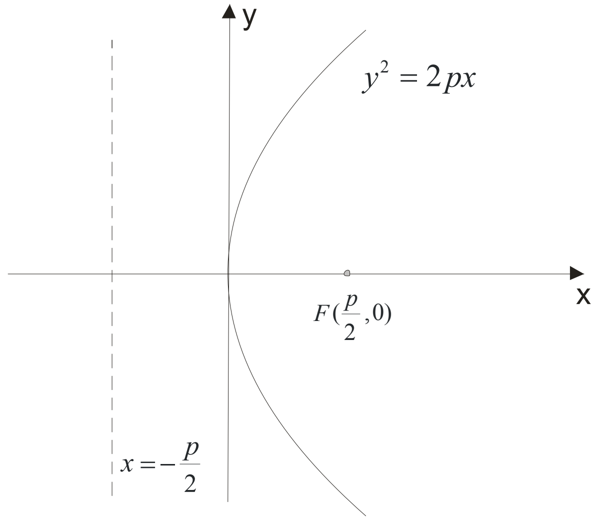
PARABLE

Parabola is the set of points in the plane with attribute that distance of each point from a constant point (focus) is the same as distance from the point of a permanent line (directrix).



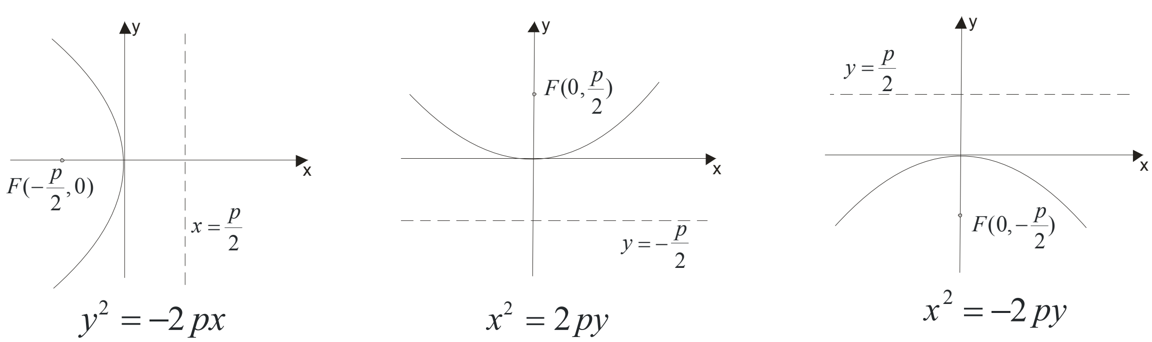
$F(\frac{p}{2},0)$ - this is the focus point of the parabola

Line $x = -\frac{p}{2}$ is the directrix of the parabola , $(x + \frac{p}{2} = 0)$

Distance from point $F(\frac{p}{2},0)$ to directrix we will marked with ***p*** (the parameter of the parabola).

The equation is: $\boxed{y^2 = 2px}$

Of course, this parabola is the most studied, but here is some other parabolas:



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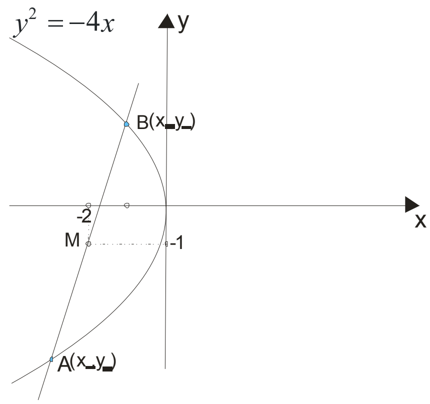
Example 1.

We have parabola $y^2 = -4x$. *Through its point M (-2, -1) set a chord that is the point halved .*

Solution:

Draw a picture to analyze the problem:



Point M in mid-long AB and must be :

$$\frac{x_1 + x_2}{2} = -2 \rightarrow x_1 + x_2 = -4$$

$$\frac{y_1 + y_2}{2} = -1 \rightarrow y_1 + y_2 = -2$$

Points A and B belong to the parabola, and their coordinates can be changed instead x and y in equation:

$$A(x_1, y_1) \in y^2 = -4x \rightarrow y_1^{\,2} = -4x_1$$

$$B(x_2, y_2) \in y^2 = -4x \rightarrow y_2^{\,2} = -4x_2$$

In this way we get 4 equations with 4 unknowns. We ask for the coordinates of points A and B, but smarter is to find the direction of line which passes through AB .

$$k = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y_1^{\,2} = -4x_1$$

$$\underline{y_2^{\,2} = -4x_2}$$

$$y_2^{\,2} - y_1^{\,2} = -4x_2 - (-4x_1)$$

$(y_2 - y_1)(y_2 + y_1) = -4x_2 + 4x_1$ we use that $y_2 + y_1 = -2$

$$(y_2 - y_1)(-2) = -4(x_2 - x_1)$$

$$\frac{y_2 - y_1}{x_2 - x_1} = 2$$

Now the equation of line through a point *M(-2,-1)*

$$y - y_0 = k(x - x_0)$$

$$y - (-1) = 2(x - (-2))$$

$$y + 1 = 2x + 4$$

$$y = 2x + 3$$

Line and parabola

Similarly as in the circle , ellipse and hyperbola, to determine the mutual position of line and parabola , solve the system of equations:

$y = kx + n$ and $y^2 = 2px$

- If the system has no solution, then the line and the parabola is not cut, that is $p < 2kn$
- If the system has two solutions, then line cut parabola in two points $p > 2kn$
- If the system has one solution, line is tangent, and satisfies the **contact condition:** $p = 2kn$

Note;

If we seek an tangent line at a given point (x_0, y_0) which belongs to theparabola, we have formula:

$$y \cdot y_0 = p(x + x_0)$$

Example 2.

Find a point on $y^2 = 16x$ *whose tangent line is tilted to an angle of 135 degrees (* 135^0 *)to the x-axis*

Solution:

Mark that point (x_0, y_0). Tangent will be $y \cdot y_0 = p(x + x_0)$. From parabola, we have:

2p = 16

p = 8

$$y \cdot y_0 = 8(x + x_0)$$
$$y \cdot y_0 = 8x + 8x_0$$
$$y = \frac{8}{y_0}x + \frac{8x_0}{y_0}$$
$$k = \frac{8}{y_0}$$

We found line direction, and how:

$$k = tg\alpha$$
$$k = tg135^0$$
$$k = -1$$

Then is:

$$k = \frac{8}{y_0}$$
$$-1 = \frac{8}{y_0} \to y_0 = -8$$

Substituting this value in the equation of parabola ,we have

$$y^2 = 16x$$
$$(-8)^2 = 16x$$
$$16x = 64$$
$$x = 4$$

Therefore, the requested point is (4, -8)

Example 3.

Write the equation of tangent parabola $y^2 = 12x$ *if it is parallel with line* $3x - y - 4 = 0$

Solution:

How is our line parallel to the given, they have the same k (condition parallels)

$$3x - y - 4 = 0$$
$$y = 3x - 4$$
$$k = 3$$

For now, we have $y = 3x + n$, and n will be found by using **contact condition:** $p = 2kn$

From parabola is $y^2 = 12x \to 2p = 12 \to p = 6$

$$p = 2kn$$
$$6 = 2 \cdot 3n$$
$$6 = 6n$$
$$n = 1$$

Solution is: $y = 3x + 1$

Example 4.

Write the equation of the common tangent line for $y^2 = 4x$ *and* $x^2 + y^2 - 2x - 9 = 0$

Solution:

$$x^2 + y^2 - 2x - 9 = 0$$
$$x^2 - 2x + y^2 - 9 = 0$$
$$\underline{x^2 - 2x + 1} - 1 + y^2 - 9 = 0$$
$$(x-1)^2 + y^2 = 10 \to p = 1, q = 0, r^2 = 10$$

Search tangent: $y = kx + n$.

It must satisfy contact condition with parabola and with circle.
From parabola we have $y^2 = 4x$, then 2p = 4 , so **p = 2**

$$p = 2kn$$

$$2 = 2kn$$

$$\boxed{kn = 1}$$

For circle:

$$r^2(k^2+1) = (kp - q + n)^2$$

$$10(k^2+1) = (1 \cdot k + n)^2$$

$$\boxed{10(k^2+1) = (k+n)^2}$$

We have two equations with two unknowns, solve the system:

$$\boxed{10(k^2+1) = (k+n)^2}$$

$$\boxed{kn = 1}$$

$$n = \frac{1}{k}$$

$$10(k^2+1) = (k+\frac{1}{k})^2$$

$$10(k^2+1) = (\frac{k^2+1}{k})^2$$

$$10(k^2+1) = \frac{(k^2+1)^2}{k^2}$$

$$10k^2 = k^2 + 1$$

$$9k^2 = 1$$

$$k^2 = \frac{1}{9} \rightarrow k_1 = \frac{1}{3}, k_2 = -\frac{1}{3}$$

Let's go back to find n:

$$kn = 1$$

$$k_1 = \frac{1}{3} \rightarrow n_1 = 3$$

$$k_2 = -\frac{1}{3} \rightarrow n_2 = -3$$

Tangent line equations are:

$$t_1: \quad y = \frac{1}{3}x + 3$$

$$t_2: \quad y = -\frac{1}{3}x - 3$$

Why do we appear two solutions?

If we draw the problem, we see that it is obvious ..